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PARAMETRIC ANALYSIS OF LARGE AMPLITUDE FREE VIBRATIONS OF A SUSPENDED CABLE

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Abstract—Partial differential equations of motion suitable to study moderately large free oscillations of an elastic suspended cable are obtained. An integral procedure is used to eliminate the spatial dependence and to reduce the problem to one ordinary differential equation which shows quadratic and cubic nonlinearities. The frequency-amplitude relationship for symmetric and antisymmetric vibration modes is studied and a numerical investigation is performed to describe the nonlinear phenomenon in a large range of values of the cable sag-to-span ratio. Softening and hardening behaviour is evidenced dependent on both the cable properties and the amplitude of oscillation.

1. INTRODUCTION

Natural oscillations of an elastic suspended cable in the linearized theory are the subject of several recent studies [1–4]. Actual oscillations occurring in technical applications of cables, e.g. in overhead transmission lines, are however of large amplitude, of the order of magnitude of the sag, and therefore they can be conveniently described through a nonlinear continuum theory only. Though similar to a certain extent to nonlinear effects arising in the classical problem of perfectly or nearly taut string, well studied in the literature[5, 6], the nonlinear phenomena of cable dynamics, already observed in earlier works[7], did not receive comparable attention; it must be noticed however that the differential problem is more complex in the latter case due to the initial curvature of the system which causes both quadratic and cubic nonlinearities.

Some recent works were devoted to the study of nonlinear oscillations of cables. In Ref. [8] a simple mechanical model is considered to analyse in-plane and out-of-plane vibrations; in Ref. [9] a model more refined as regards the kinematical description of the system—though simple again—is utilized to study planar oscillations only.

In the present work, after recalling the general theory of the finite dynamics of cable about the nonlinear static equilibrium configuration, equations of motion suitable to study moderately large oscillations of the system are obtained.

The important aspects of dependence of cable vibration frequency and temporal law on the amplitude are studied for planar symmetric and antisymmetric vibration modes. The parametric investigation developed completely describes the nonlinear phenomenon in the whole range of practical interest of the cable properties, with values of the sag-to-span ratio from nearly zero to about 1/8. In this range planar oscillations are stable except the case when the in-plane linear frequency is twice the out-of-plane one. The region of resonance is studied in Ref. [10].

The modification of the frequency crossover phenomenon characteristic of the linearized theory of elastic cable[2–4], caused by the nonlinear frequency-amplitude dependence is discussed as well.

An integral procedure is used to eliminate the spatial dependence and an assumed mode technique—already used for cables in Ref. [9]—is referred to; the solution of the ensuing ordinary differential equation is pursued through numerical integration.

2. DEVELOPMENT OF THE EQUATIONS OF MOTION

Consider an elastic heavy cable suspended between two fixed supports. Three different configurations are distinguished: the natural configuration $\mathscr{C}^0$ occupied by the cable with zero stress and strain fields, the initial deformed configuration $\mathscr{C}^I$ of static equilibrium under dead load, the varied configuration $\mathscr{C}^V$ occupied during the dynamic phenomenon. The characteristics of the cable in $\mathscr{C}^0$ are l_C^0, length, m^0, mass per unit length, A, area of the normal section, E, elastic material modulus. A curvilinear abscissa s along the cable and an orthogonal coordinate system $0x_i$ are introduced.

The total deformation of cable consists of an initial and an additional deformation. By assuming the Lagrangian strain as the strain measure and the initial configuration as the reference one for the dynamic displacement $u_i = x_i^V - x_i^I$, it follows:

$$\epsilon^V = \epsilon^I + \epsilon\left(\frac{ds^I}{ds^0}\right)^2$$

where

$$\epsilon = \frac{dx_i^I}{ds^I}\frac{\partial u_i}{\partial s^I} + \frac{1}{2}\frac{\partial u_i}{\partial s^I}\frac{\partial u_i}{\partial s^I} \quad (i = 1, 2, 3)$$

and the nonlinear equations of free motion read[11]:

$$\frac{\partial}{\partial s^I}\left\{T^I\frac{\partial u_i}{\partial s^I} + E^*A^I\left(\frac{dx_i^I}{ds^I} + \frac{\partial u_i}{\partial s^I}\right)\left(\frac{dx_j^I}{ds^I}\frac{\partial u_j}{\partial s^I} + \frac{1}{2}\frac{\partial u_j}{\partial s^I}\frac{\partial u_j}{\partial s^I}\right)\right\} = m^I\ddot{u}_i \quad (i, j = 1, 2, 3). \qquad (1)$$

In eqn (1) T^I is the cable tension in the initial configuration and $m^I = m^0(ds^0/ds^I)$, $E^*A^I = EA\,(ds^I/ds^0)^3$ are the values of mass and axial rigidity of unit cable element in the reference configuration, which account for the nonlinear deformation $\mathscr{C}^0 \to \mathscr{C}^I$.

As far as the cable tension T^V in the dynamic configuration is concerned, it is obtained from the relation:

$$T^V = \mathscr{T}^V\left(\frac{ds^V}{ds^0}\right) = (T^I + E^*A^I\epsilon)\left(\frac{ds^V}{ds^I}\right)$$

where $\mathscr{T}^V$ is the tension in $\mathscr{C}^V$ expressed in terms of the Kirchoff stress.

For a suspended cable (Fig. 1) used in a overhead transmission line usually the sag-to-span ratio d/l is about 1/20. In these conditions and for d/l up to 1/8, the static equilibrium configuration of the cable can be described through a parabola:

$$y = 4d\left[\frac{x}{l} - \left(\frac{x}{l}\right)^2\right]$$

This entails that $(ds - dx)/dx$ and H/EA are $0[(d/l)^2]$, H being the horizontal component of the initial tension, while the dynamical displacement components are respectively $u = 0[\epsilon\, d^2/l]$ and $v = 0[\epsilon\, d]$ where ϵ is a small parameter of the order of the amplitude.

Retaining terms of order up to $(d/l)^3$, and restricting attention to planar (xy) oscillations of cable, the equations of motion read:

$$\frac{\partial}{\partial x}\left\{EA\left[\frac{\partial u}{\partial x} + \frac{dy}{dx}\frac{\partial v}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2\right]\right\} = m\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial}{\partial x}\left\{H\frac{\partial v}{\partial x} + EA\left(\frac{dy}{dx} + \frac{\partial v}{\partial x}\right)\left[\frac{\partial u}{\partial x} + \frac{dy}{dx}\frac{\partial v}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2\right]\right\} = m\frac{\partial^2 v}{\partial t^2} \qquad (2)$$

where suffixes "I" and "0" were omitted.

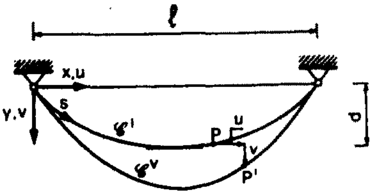

Fig. 1. Cable configurations.

The ordering of terms followed corresponds to assume $ds' \simeq dx'$, $T' \simeq H$, $H/EA \ll 1$ and $du/dx' \ll 1$ in eqns (1). Equations (2) still contain quadratic and cubic terms.

To study the influence of these terms on a given vibration mode, a dimensionless form of the equations of motion is obtained through the positions:

$$\tilde{x} = x/l,\ \tilde{u} = u/d,\ \tilde{v} = v/d,\ \tilde{t} = \omega t \tag{3}$$

where ω is the linear frequency of the cable vibration eigenmode. Accounting for the initial configuration and omitting the tilde for the sake of simplicity it follows:

$$\frac{\partial}{\partial x}\left\{\mu\left[\frac{\partial u}{\partial x} + 8\nu\left(\frac{1}{2} - x\right)\frac{\partial v}{\partial x} + \frac{1}{2}\nu\left(\frac{\partial v}{\partial x}\right)^2\right]\right\} = \pi^2 n^2\left(\frac{\omega}{\omega_0}\right)^2\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial}{\partial x}\left\{\frac{\partial v}{\partial x} + \mu\left[8\nu\left(\frac{1}{2} - x\right) + \nu\frac{\partial v}{\partial x}\right]\left[\frac{\partial u}{\partial x} + 8\nu\left(\frac{1}{2} - x\right)\frac{\partial v}{\partial x} + \frac{1}{2}\nu\left(\frac{\partial v}{\partial x}\right)^2\right]\right\} = \pi^2 n^2\left(\frac{\omega}{\omega_0}\right)^2\frac{\partial^2 v}{\partial t^2} \tag{4}$$

where $\mu = EA/H$, $\nu = d/l$ and the frequency of the corresponding mode of the taut string with the same tension H of the cable:

$$\omega_0 = n\frac{\pi}{l}\sqrt{\frac{H}{m}} \quad (n = 1, 2, 3, \ldots\ldots) \tag{5}$$

has been introduced.

3. DESCRIPTION OF THE DISCRETE MODEL

A discrete model of the continuum system is obtained by assuming a shape function for the dynamic displacement. Solution to eqns (4) is thus sought in the form:

$$u(x, t) = g(x)q(t),\ v(x, t) = f(x)q(t) \tag{6}$$

where $q(t)$ is an unknown time function and $g(x), f(x)$ are the components of the assumed eigenfunction of the linearized problem associated with system (4) in which the longitudinal inertia forces have been neglected. These eigenfunctions are furnished in Ref. [2]; if normalized in such a way that the maximum amplitude of the vertical displacement component is equal unity and being $\nu^2 \ll 1$, they read as follows:

—*symmetric modes*

$$g_i(x) = 8\nu B_i\left\{\frac{(\beta_i l)^2}{\lambda^2} - \frac{1}{2}(1 - 2x)\left(1 - \tan\frac{\beta_i l}{2}\sin\beta_i lx - \cos\beta_i lx\right) - \frac{1}{\beta_i l}\left[\beta_i lx - \tan\frac{\beta_i l}{2}(1 - \cos\beta_i lx) - \sin\beta_i lx\right]\right\} \tag{7}$$

$$f_i(x) = B_i\left\{1 - \tan\frac{\beta_i l}{2}\sin\beta_i lx - \cos\beta_i lx\right\} \tag{8}$$

in which $B_i = \cos(\beta_i l)/(2)/[\cos(\beta_i l)/(2) + (-1)^i]$ and $\beta_i l$ are roots of the equation:

$$\tan\frac{\beta l}{2} = \frac{\beta l}{2} - \frac{4}{\lambda^2}\left(\frac{\beta l}{2}\right)^3 \tag{9}$$

where

$$\lambda^2 = 64\,\mu v^2 \tag{10}$$

and the frequencies follow from:

$$\omega_i = \beta_i\sqrt{\frac{H}{m}} \tag{11}$$

with $i = 1, 2, 3, \ldots$. being the order of the ith symmetric mode.

—*antisymmetric modes*

$$g_i(x) = -4v\left\{(1-2x)\sin 2i\pi x + \frac{1}{i\pi}(1-\cos 2i\pi x)\right\} \tag{12}$$

$$f_i(x) = \sin 2i\pi x \tag{13}$$

and the frequencies are given by:

$$\omega_i = \frac{2i\pi}{l}\sqrt{\frac{H}{m}} \tag{14}$$

with $i = 1, 2, 3, \ldots$. being the order of the ith antisymmetric mode.

The integral formulation of the equations of motion reads:

$$\int_0^1\left\{G(x)\frac{\partial\delta u}{\partial x} + F(x)\frac{\partial\delta v}{\partial x} + \pi^2 n^2\left(\frac{\omega}{\omega_0}\right)^2(\ddot{u}\delta u + \ddot{v}\delta v)\right\}\mathrm{d}x = 0 \tag{15}$$

where $G(x)$ and $F(x)$ are the bracketed expressions { } in eqns (4), and δu, δv are virtual variations of displacement, with $(^{\bullet}) = (\partial/\partial t)$.

By using relationships (6), an ordinary differential equation in the normal coordinate q is obtained:

$$\ddot{q} + q + c_2 q^2 + c_3 q^3 = 0. \tag{16}$$

The coefficients c_2 and c_3 of the quadratic and cubic terms are as follows:

$$c_2 = \frac{3}{2}\mu v\frac{I_2}{I_T + \mu I_1}, \quad c_3 = \frac{1}{2}\mu v^2\frac{I_3}{I_T + \mu I_1} \tag{17}$$

where the following positions were made, with $(') = (\mathrm{d}/\mathrm{d}x)$:

$$I_1 = \int_0^1\left[g' + 8v\left(\frac{1}{2} - x\right)f'\right]^2\mathrm{d}x \qquad I_T = \int_0^1 f'^2\mathrm{d}x$$
$$I_2 = \int_0^1 f'^2\left[g' + 8v\left(\frac{1}{2} - x\right)f'\right]\mathrm{d}x \qquad I_3 = \int_0^1 f'^4\mathrm{d}x. \tag{18}$$

The eigenfunctions adopted are such that the function $[g' + 8v(\frac{1}{2} - x)f']$, which apart from term $EA\,q(t)$ represents the linear part of the dynamic variation ΔT of cable tension, is a constant. For symmetric modes it results:

$$C_i = B_i \frac{(\beta_i l)^2}{8\nu\mu} \tag{19}$$

while for antisymmetric modes it is zero; so in the latter case c_2 vanishes and the equation of motion shows cubic term only as the actual mechanical problem requires.

As far as the variation of cable tension in the nonlinear motion is concerned it results:

—*symmetric modes*

$$\left(\frac{\Delta T(x)}{H}\right)_i = \frac{1}{8} B_i(\beta_i l)^2 q + \frac{1}{128}\lambda^2 B_i^2(\beta_i l)^2 \left[\sin \beta_i l x - \tan\frac{\beta_i l}{2}\cos \beta_i l x\right]^2 q^2 \tag{20a}$$

—*antisymmetric modes*

$$\left(\frac{\Delta T(x)}{H}\right)_i = \frac{1}{32} i^2\pi^2\lambda^2 q^2 \cos^2 2i\pi x. \tag{20b}$$

The analysis of the eqns (16)–(20) shows that due to the nondimensionalisation with respect to sag adopted herein, the nonlinear oscillations of a suspended cable are governed by the unique parameter λ^2, defined by eqn (10), which collects its geometrical and mechanical properties, just as in the linearized theory[2, 4].

To study the motion of the system, consider the first integral of eqn (16):

$$h = \frac{1}{2}\dot{q}^2 + F(q) \tag{21}$$

which is the energy of the conservative discrete model; the elastic potential energy $F(q)$ is given by:

$$F(q) = \frac{1}{2}q^2 + \frac{1}{3}c_2 q^3 + \frac{1}{4}c_3 q^4. \tag{22}$$

Referring to the first symmetric mode of oscillation the analysis of the F vs q diagram reported in Fig. 2 together with the load—deflection curve gives information about the effectiveness of the discrete model considered in representing the mechanical behaviour of the actual system. $F(q)$ can exhibit three stationary points given by:

$$q = 0, \quad q = (-c_2 \pm \sqrt{c_2^2 - 4c_3})/2c_3. \tag{23}$$

The first root corresponds to the initial static equilibrium configuration of the cable (F); the other two, if real, define the two equilibrium configurations above the horizontal line passing through the supports, one stable (H) and one unstable (G) as in the classical two pin-ended bars system. In Fig. 2 the symmetrical natural configurations of the cable (A) and (B) are also evidenced as well as the horizontal one (C). The two configurations (G) and (H) would be possible for the cable if it was capable to maintain its geometrical shape in all positions but they are not actually admissible.

As far as the present discrete model is concerned, the quantity $(c_2^2 - 4c_3)$ is negative in the whole range of the sag-to-span cable ratio (see Table 1) so that $F(q)$ always increases versus $|q|$ (dashed curve in Fig. 2c). This curve correctly describes the behaviour of initially prestressed cables which actually show one stationary point only, corresponding to the initial state. For non-prestressed cables the $F(q)$ diagram does not have upward curvature everywhere and thus symmetric oscillations of moderate amplitude must be considered in order to maintain cable in tension; in this range which will be analyzed in Section 4, the discrete model considered is still representative. In both cases the motion about the initial

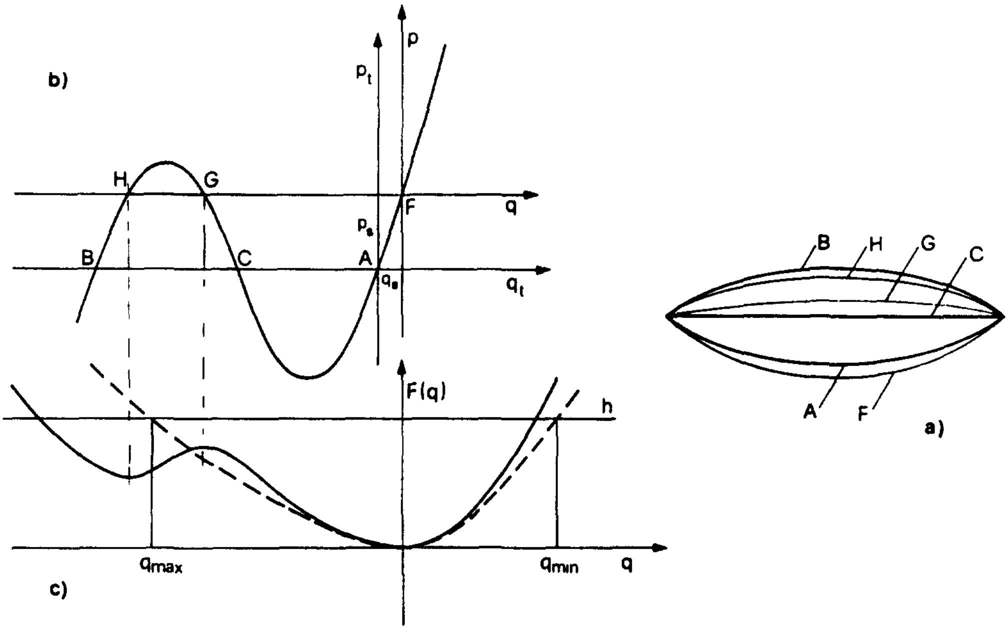

Fig. 2. Load-deflection curve (b) and potential curve (c) of a cable with constant geometrical shape (a).

Table 1. c_2 and c_3 coefficients for the 1st and 2nd symmetric mode and the 1st antisymmetric mode

	1st sym		1st antisym		2nd sym	
λ^2	c_2	c_3	c_2	c_3	c_2	c_3
0.512	0.0586	0.0281	0	0.1184	0.0260	0.2670
4.352	0.3769	0.1719	0	1.0067	0.1831	2.3182
39.47	1.2337	0.7610	0	9.1321	2.7586	27.774
81.92	2.5510	5.0614	0	18.949	12.617	134.14
117.7	3.2698	13.804	0	27.240	54.116	1541.3
230.4	3.3046	51.799	0	53.295	131.16	10496.

configuration is unsymmetrical with greater amplitude for upward displacements of the cable. For the amplitude of oscillation at a given energy level h reference will be made to the average value $q_m = (|q_{max}| + q_{min})/2$.

The nonlinear frequency of oscillation Ω could be calculated from the temporal law of the motion $t \rightarrow q(t)$, obtained from eqn (21):

$$\Omega = \pi \Big/ \int_{q_{min}}^{q_{max}} \frac{dq}{[2h - F(q)]^{1/2}}.$$

Here a numerical integration of eqn (16) by the Numerov method using a predictor-corrector procedure will be effected. The initial conditions of the motion characterized by a prescribed q_m value are obtained by means of eqn (22).

4. FREQUENCY-AMPLITUDE RELATIONSHIP FOR THE SYMMETRIC AND ANTISYMMETRIC MODES

A numerical investigation of the frequency-amplitude relationship for cable oscillating in one mode was made by varying its mechanical and geometrical properties through the parameter λ^2 in a large range of amplitude values. With technical values of the mechanical properties ($\mu = 300 \div 500$) the values of (d/l) considered rise up to about 1/8.

The first and second symmetric mode as well as the first antisymmetric mode are studied. The latter has one nodal point in the whole range considered while the shape of the two symmetric modes changes from zero to two nodal points and from two to four nodal points respectively according to the crossover phenomenon [2–4], as shown in Fig. 3.

Analysis of coefficients c_2 and c_3 of the equation of motion for the three modes (see Table 1) shows that an increase in λ^2 makes the problem more strongly nonlinear and that nonlinearities are higher for the antisymmetric modes.

The ratio Ω/ω between the nonlinear and linear frequency of oscillation for the three modes is plotted vs the nondimensionalized average amplitude q_m in Figs. 4–6.

According to zeroing of c_2 and positiveness of c_3 the nonlinear behaviour of the antisymmetric mode is always hardening (Fig. 4); the corrections of frequency with respect to the linear one rise up to quite high values.

The trend of the corresponding curves for the symmetric modes is much more complex varying considerably with both the properties of the cable and the amplitude of oscillation;

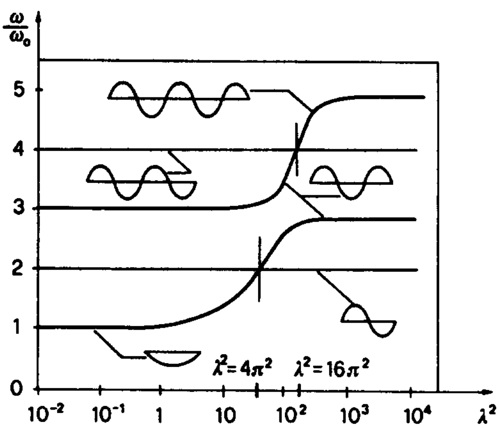

Fig. 3. Crossover phenomenon for linear elastic suspended cable and vibration shapes.

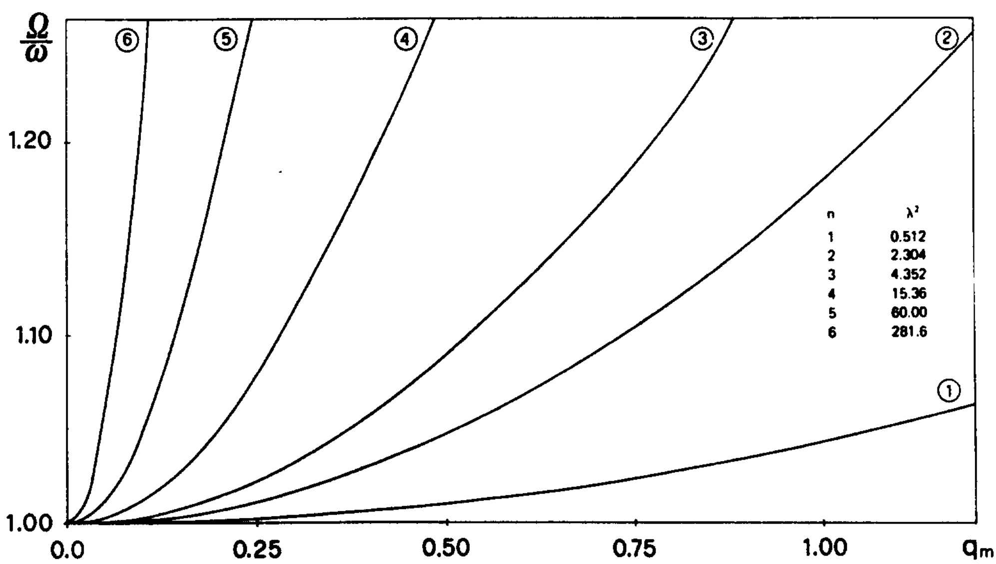

Fig. 4. Frequency-amplitude relationship for the first antisymmetric mode.

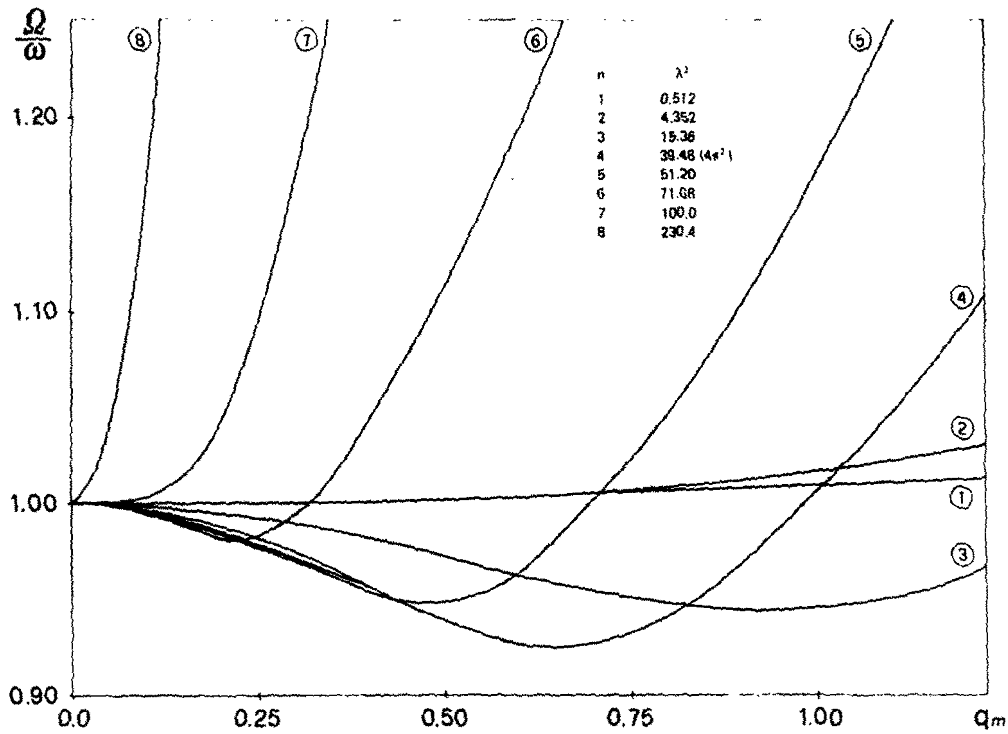

Fig. 5. Frequency-amplitude relationship for the first symmetric mode.

in this respect it is similar to that evidenced for shallow arches[12]. In a large range of low values of q_m, the dynamic behaviour is generally softening while at higher values it becomes hardening for all cables due to definite prevailing of cubic term. The transition from the softening to hardening behaviour occurs at an amplitude as higher as higher λ^2 is, till a value of this parameter is reached above which the amplitude of transition quickly diminishes up to zero.

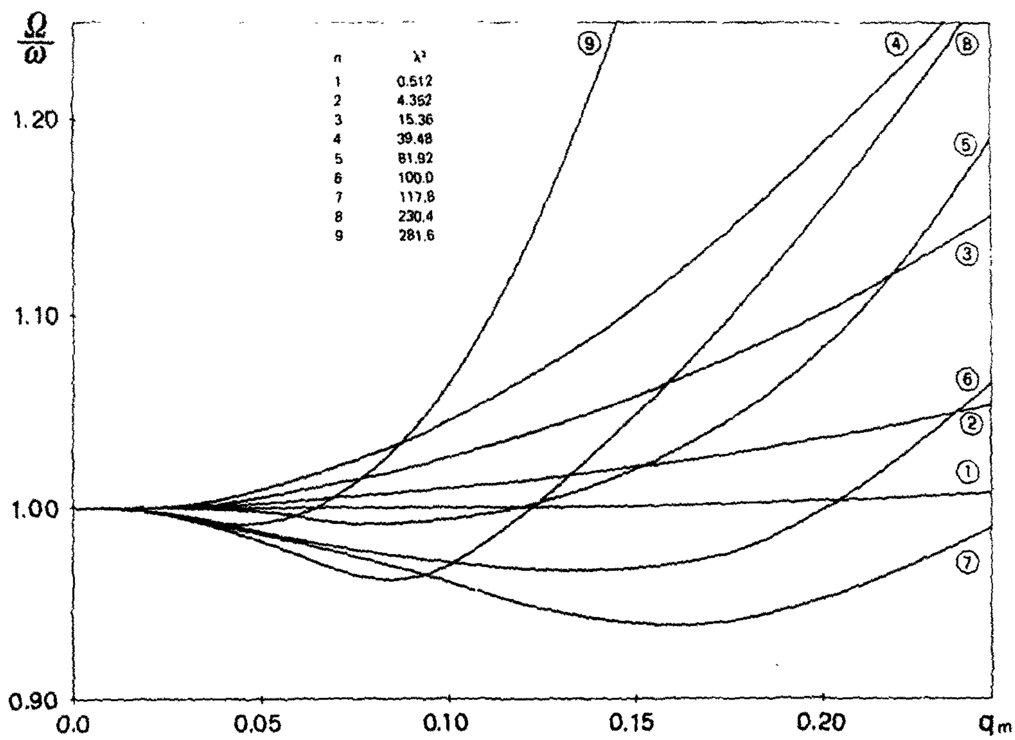

Fig. 6. Frequency-amplitude relationship for the second symmetric mode.

Instead for the lowest and highest values of λ^2, the behaviour of the cable is hardening in the whole range of amplitude. The first circumstance is consistent with the always positive nonlinear correction of frequency occurring for the taut string. The second circumstance is bound up with the modification in the shape of oscillation as λ^2 grows up (see Fig. 3), according to which the system becomes stiffer and the frequency corrections are always positive.

A comparison between Figs. 5 and 6 shows that the oscillation of cable in the second mode is more strongly nonlinear with respect to the first one, according to results obtained for beams (e.g. see Ref. [13]). The figure relevant to second mode puts into better evidence the general trend of the frequency-amplitude relationship: at first it is strictly hardening, then it becomes softening-hardening depending on the amplitude, at last it is hardening again. The modification of the behaviour from the weakly to the strongly hardening curves occurs in the range of λ^2 values about the crossover points of the linear frequency in which the shape of oscillation gradually changes (Fig. 3).

The temporal law of the motion for two cables oscillating in the first symmetric mode is plotted in Fig. 7 for three values of the amplitude. At the lowest value it is practically coincident with the linearized law for both cables. At higher values of the amplitude the shape of the law differs notably from the simple cosine function, showing a drift of the midpoint of oscillation towards the negative values of q which is stronger the slacker the cable is and the higher the amplitude is.

The values of λ^2 considered in the numerical investigation correspond to both initially prestressed cables and slack cables. Indeed by imposing the natural configuration of the cable to be the straight one between the two supports the value $\lambda^2 \simeq 2.43\,\pi^2$ is obtained as identifying the cable which separates the two classes.

As far as prestressed cables are concerned the whole range of oscillation amplitudes considered holds good, while for non-prestressed cables compressive tension occurs at a certain value of amplitude; in the latter case the model adopted in this study, which exhibits an assumed space shape during the motion, furnishes results not strictly representative of the actual phenomenon for large amplitudes. For symmetric modes, eqn (20a) was used to determine the limiting values of amplitude corresponding to positive cable tension (see Table 2). No limitations occur as regards antisymmetric modes.

5. EFFECT OF NONLINEARITIES ON THE CROSSOVER FREQUENCIES

The crossover phenomenon evidenced in the linear theory of the elastic cable, shown in Fig. 3, consists of an inversion between symmetric and corresponding antisymmetric modes in the natural order of frequencies, whose main effect is that with $\lambda^2 > 4\pi^2$ the first vibration mode is the antisymmetric one. In the nonlinear field the dependence of frequency on the amplitude of oscillation produces some modifications of this phenomenon. Restricting attention to the crossover of the first modes, one crossover point only

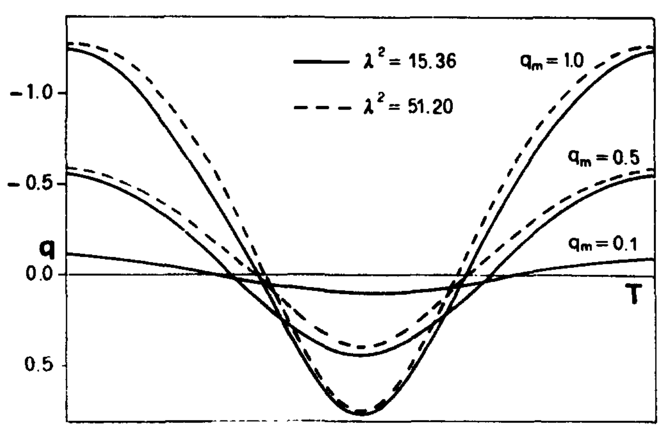

Fig. 7. Law of the motion.

Table 2. Limiting values of amplitude for non-prestressed cables

λ^2	39.47	60.01	81.92	99.99	117.7	230.4	281.6
1st sym	0.405	0.237	0.313	0.323	0.333	0.421	0.469
2nd sym	0.664	0.358	0.220	0.165	0.134	0.093	0.096

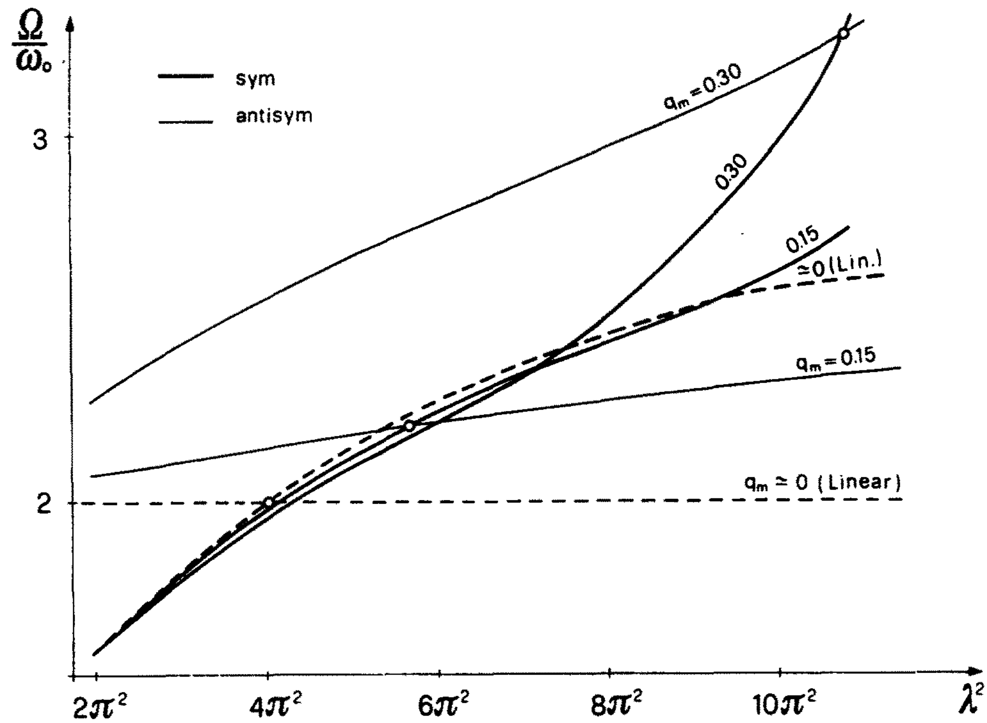

Fig. 8. Modification of the first crossover frequency due to nonlinearities.

exists in the linear theory, while nonlinearities reveal the existence of an infinite number of crossover points. Indeed for a given cable having $\lambda^2 > 4\pi^2$ there exist infinite couples of oscillations of different amplitude, one symmetric and the other antisymmetric, characterized by the same frequency: i.e. in an extended range above the linear crossover point every frequency can be considered a crossover one. Figure 8 shows the relationship Ω/ω_0 vs λ^2 in the upper neighbourhood of $\lambda^2 = 4\pi^2$, for two couples of oscillations with equal finite amplitude; two new crossover points are revealed in addition to the linear one. Within the assumption of equal oscillation amplitude, for every λ^2 value the crossover frequency Ω_c/ω_0 and the corresponding amplitude of the two oscillations can be read in Fig. 9; it shows that, depending on the amplitude, even with $\lambda^2 > 4\pi^2$ the first vibration mode can be a symmetric one, for the values of amplitude and frequency higher than those corresponding to the two curves.

6. CONCLUSIONS

Nonlinear planar free dynamics of a parabolic suspended cable are studied showing that its behaviour depends on a unique parameter collecting the geometrical and mechanical properties as in the linearized theory.

By varying this parameter the modification of the linear frequency with the motion amplitude of both prestressed and slack cables is studied. For the first symmetric mode it ranges from some percents to about twenty percents for amplitudes of the order of the

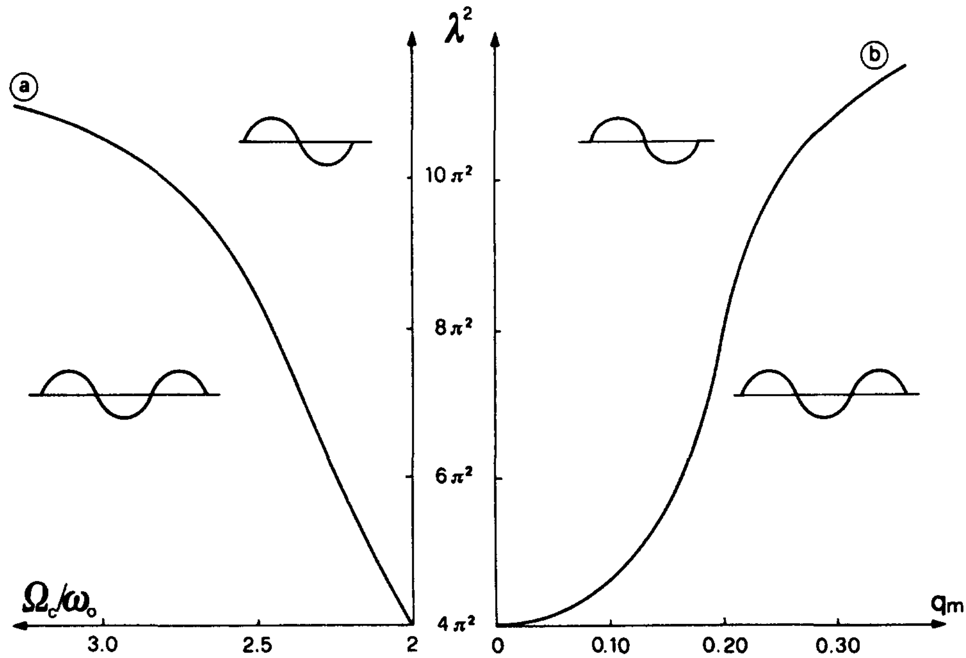

Fig. 9. Crossover frequency (a) and amplitude (b) for sym and antisym oscillations of equal amplitude.

sag; for the second sym mode and the first antisym one it is very strong at low amplitudes as well. The dynamical behaviour is either hardening or softening depending on the cable properties and the motion amplitude and the effects of nonlinearities are as stronger as the sag-to-span ratio and the mode number are higher.

Finally the modification of crossover phenomenon is examined revealing an infinite number of crossover points instead of the single one of the linear theory.

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